CHAPTER II
DIFFERENTATION IN SEVERAL VARIABLES

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Introduction to Functions of Several Variables

Definition 1

Let D be a nonempty set and that for each element x in D there corresponds a unique value $y = f(x)$ in R, then f is called a **function of** x. The set D is the **domain** of f, and R is the **range** of f. We write

$$f : D \longrightarrow R; x \longmapsto f(x).$$

In this case x is called **independent variable** and y is called **dependent variable**.

Extrema of Functions of Several Variables

Definition 52

Let $D \subset \mathbb{R}^n$ and $f : D \to \mathbb{R}$ be a function.

- The set D is said to be **convex** if every two points $x, y \in D$, the line segment $L(x, y) \subset D$.
- The function f is said to be **convex** on a convex set D if for any $x, y \in D$ and for any $t \in [0,1], f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$.
- The function f is said to be **concave** on a convex set D if for any $x, y \in D$ and for any $t \in [0,1], f((1-t)x + ty) \geq (1-t)f(x) + tf(y)$.

Theorem 53

Let $D \subset \mathbb{R}^n$ be a convex set and $f : D \to \mathbb{R}$ be a C^2 function. The function f is a convex function on D if and only if for any $x \in D$, for any $k = 1, 2, \ldots, n$, the kth principal minor of the Hessian $H_k(x) \geq 0$.

Extrema of Functions of Several Variables

Theorem 54

If $f : D \to \mathbb{R}$ is a convex function on a convex set D and $f(a)$ is a local minimum then $f(a)$ is the global minimum on D.
If $f : D \to \mathbb{R}$ is a concave function on a convex set D and $f(a)$ is a local maximum then $f(a)$ is the global maximum on D.

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Lagrange Multipliers

Theorem 55

Let $f, g_1, \ldots, g_k : D \to \mathbb{R}$ be C^1 functions where $D \subset \mathbb{R}^n$ is open and $k < n$. Suppose there is an $a \in D$ such that

$$\det \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(a) & \cdots & \frac{\partial g_1}{\partial x_k}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial x_1}(a) & \cdots & \frac{\partial g_k}{\partial x_k}(a) \end{pmatrix} \neq 0.$$

If $f(a)$ is local extremum of f subject to the constraints $g_i(x) = c_i$ for $i = 1, 2, \ldots, k$, then there exist scalars $\lambda_1, \ldots, \lambda_k$ (called **Lagrange Multipliers***) such that*

$$\nabla f(a) + \sum_{i=1}^{k} \lambda_i \nabla g_i(a) = 0.$$

Extrema of Functions of Several Variables

Theorem 46 (A necessary condition for extremum)

If f has a local extremum at a on an open region D, then a is a critical point of f.

Note that the converse of the theorem above is not true in general. That is, a critical point does not yield a local extremum.

Definition 47 (Quadratic form)

Let $b_{ij} \in \mathbb{R}$ such that $b_{ij} = b_{ji}$ and $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$.

1. A **quadratic form** in $h_1, \ldots, h_n$ is a function defined by

$$Q(h_1, \ldots, h_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} h_i h_j.$$

This quadratic form can be written in term of matrices as

$$Q(h) = (h_1 \ldots h_n) \begin{pmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \ldots & b_{nn} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = h^t B h$$

where $h^t = \begin{pmatrix} h_1 & \ldots & h_n \end{pmatrix}$ and $B = (b_{ij})_{n \times n}$ is a symmetric matrix.

2. The quadratic form Q (and also symmetric matrix B) is said to be **positive definite** if $Q(h) > 0$ for all $h \neq 0$ and **negative definite** if $Q(h) < 0$ for all $h \neq 0$.

Extrema of Functions of Several Variables

Theorem 48

Let $A = (a_{ij})_{n \times n}$ be a symmetric matrix. Then the matrix A is positive definite if and only if all kth principal minors $A_k > 0$ for $k = 1, 2, \ldots, n$.

Theorem 49

Let$A = (a_{ij})_{n \times n}$ be a symmetric matrix. Then A is positive definite if and only if it can be reduced to upper triangular form using only elementary row operations $E_{i,j}(\lambda)$ and the diagonal elements of resulting matrix are greater than zero.

Extrema of Functions of Several Variables

Theorem 50

Let $D \subset \mathbb{R}^n$ be open and $f : D \to \mathbb{R}$ be a function. If all second-order partial derivatives of f exist at $a \in D$ and $d^{(2)} f_a(h) > 0$ for all $h \neq 0$, then there is a $\lambda > 0$ such that $d^{(2)} f_a(x) \geq \lambda \|x\|^2$ for all $x \in \mathbb{R}^n$.

Theorem 51 (The Second Derivative Test)

Let $D \subset \mathbb{R}^n$ be open containing a and $f : D \to \mathbb{R}$ satisfy $\nabla f(a) = 0$. Suppose further that all second-order partial derivatives of f exist on D and continuous at a.

- *If the quadratic form $Q = d^{(2)} f_a(h)$ is positive definite, then $f(a)$ is a local minimum of f.*
- *If the quadratic form $Q = d^{(2)} f_a(h)$ is negative definite, then $f(a)$ is a local maximum of f.*
- *If the quadratic form $Q = d^{(2)} f_a(h)$ takes on both positive and negative values for $h \in \mathbb{R}^n$, then a is a saddle point of f.*

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Extrema of Functions of Several Variables

Definition 41

Let $D \subset \mathbb{R}^n$, $f : D \to \mathbb{R}$ be a function and $a \in D$.

- $f(a)$ is called a **local minimum** of f if there is an $r > 0$ such that $f(a) \leq f(x)$ for all $x \in B_r(a) \cap D$.
- $f(a)$ is called a **local maximum** of f if there is an $r > 0$ such that $f(a) \geq f(x)$ for all $x \in B_r(a) \cap D$.
- $f(a)$ is called a **local extremum** of f if there $f(a)$ is a local minimum or a local maximum of f.
- $f(a)$ is called a **global minimum** of f on D if $f(a) \leq f(x)$ for all $x \in D$.
- $f(a)$ is called a **global maximum** of f on D if $f(a) \geq f(x)$ for all $x \in D$.
- $f(a)$ is called a **global extremum** of f on D if $f(a)$ is a global minimum or a global maximum of f.

Extrema of Functions of Several Variables

Theorem 42

If $D \subset \mathbb{R}^n$ is closed and bounded and $f : D \to \mathbb{R}$ is a continuous function, then f must have both a global maximum and a global minimum somewhere on D.

Theorem 43

Let $D \subset \mathbb{R}^n$ be open and $f : D \to \mathbb{R}$ be a function. If the first-order partial derivatives of f exist at $a \in D$ and $f(a)$ is a local extremum of f, then $\nabla f(a) = 0$.

Extrema of Functions of Several Variables

Definition 44

Let f be defined on an open region D containing a. The point a is a **critical point** of f if one of the following is true.

1. $f_{x_i}(a) = 0$, for all $i = 1, 2, \ldots, n$.
2. $f_{x_i}(a)$ does not exist for some $i \in \{1, 2, \ldots, n\}$.

Definition 45

Let $D \subset \mathbb{R}^n$ be open and $f : D \to \mathbb{R}$ be differentiable at $a \in D$. Then a is called a **saddle point** of f if $\nabla f(a) = 0$ and there is an $r_0 > 0$ such that given any $0 < \rho < r_0$ there are points $x, y \in B_\rho(a)$ which satisfy $f(x) < f(a) < f(y)$.

Tangent planes and normal lines

So far, you have represented surfaces in space primarily by equations of the form

$$z = f(x, y)$$

In the development to follow, however, it is convenient to use the more general representation $F(x, y, z) = 0$. For a surface S given by $z = f(x, y)$, you can convert to the general form by defining F as

$$F(x, y, z) = f(x, y) - z.$$

Because $f(x, y) - z = 0$, you can consider S to be the level surface of F given by

$$F(x, y, z) = 0.$$

Tangent planes and normal lines

Let S be a surface given by $F(x, y, z) = 0$ and let $P(x_0, y_0, z_0)$ be a point on S. Let C be a curve S on through P that is defined by the vector-valued function

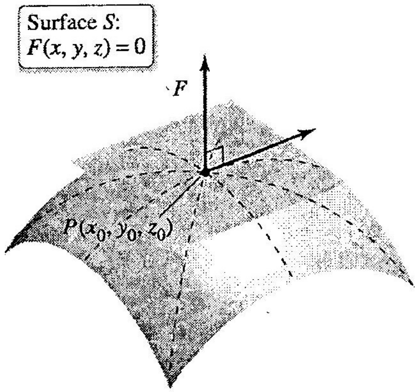
$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

Then, for all t

$$F(x(t), y(t), z(t)) = 0.$$

If F is differentiable and $x'(t), y'(t)$ and $z'(t)$ all exist, then we have

$$F'(t) = F_x(x, y, z)x'(t) + F_y(x, y, z)y'(t) + F_z(x, y, z)z'(t) = 0$$



Tangent planes and normal lines

At point $P(x_0, y_0, z_0)$, the equivalent vector form is

$$\nabla F(x_0, y_0, z_0).\mathbf{r}'(t_0) = 0.$$

This result means that the gradient at P is orthogonal to the tangent vector of every curve on S through P. So, all tangent lines on S lie in a plane that is normal to $\nabla F(x_{0,0}, z_0)$ and contains P, as shown in the Figure.

Definition 38

Let F be differentiable at the point $P(x_0, y_0, z_0)$ on the surface given by $F(x, y, z) = 0$ such that $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$.

- The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the **tangent plane to S at P.**
- The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called the **normal line to S at P.**

Tangent planes and normal lines

Theorem 39 (Equation of Tangent plane)

If F is differentiable at (x_0, y_0, z_0) then an equation of the tangent plane to the surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Theorem 40

If the surface given by equation $z = f(x, y)$, then an equation of tangent line to the the surface at the point (x_0, y_0, z_0) is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Hessian Matrix

Definition 34 (Principal Minor)

Let $A = (a_{ij})_{n\times n}$ be a square matrix. The determinant $A_k = \det(a_{ij})_{k\times k}$ is called the kth principal minor of the $n\times n$ matrix.

Definition 35 (The Hessian of a Function)

Let $D \subset \mathbb{R}^n$ and $f : D \to \mathbb{R}$ be a function having second-order partial derivatives $\frac{\partial^2 f}{\partial x_j \partial x_i}$. The **Hessian** of f is the matrix whose (i,j) entry is $f_{x_i x_j}$. That is,

$$H_f(a) = \left(\frac{\partial^2 f}{\partial x_j \partial x_i}(a)\right)_{n\times n}$$

We call $H_s = \det\left(\frac{\partial^2 f}{\partial x_j \partial x_i}\right)_{s\times s}$, sth principal minor of H_f for $s = 1, 2, \ldots, n$.

Higher Order Differential

Definition 36 (Higher Order Differential)

Let $D \subset \mathbb{R}^n$ and $f : D \to \mathbb{R}$ be a function. Suppose that the partial derivatives of f order $k-1$ exist on D. If each $(k-1)$th order partial derivative of f is differentiable at $a \in D$. Let $h = (h_1, \ldots, h_n)$. We call the kth **differential of** f **at** a is the expression

$$d^{(k)} f_a(h) = \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1} \ldots \partial x_{i_k}}(a) h_{i_1} \ldots h_{i_k}.$$

Theorem 37 (Taylor's formula)

Let $D \subset \mathbb{R}^n$ be open, $a, x \in D$, and $f : D \to \mathbb{R}$ be a function, and suppose that the partial derivatives of f order $k-1$ exist on D. If each $(k-1)$th order partial derivative of f is differentiable on D and the line segment $L(a,x) = \{(1-t)a + tx, 0 \le t \le 1\} \subset D$, then there is a point $c \in L(a,x)$ such that

$$f(x) = f(a) + \sum_{j=1}^{k-1} \frac{1}{j!} d^{(j)} f_a(h) + \frac{1}{k!} d^{(k)} f_c(h)$$

$$f(x) = f(a) + df_a(h) + \frac{1}{2} h^t H_f(a) h + \sum_{j=3}^{k-1} \frac{1}{j!} d^{(j)} f_a(h) + R_k$$

where $d^{(j)} f_a(h) = \sum_{i_1=1}^{n} \cdots \sum_{i_j=1}^{n} \frac{\partial^j f}{\partial x_{i_1} \ldots \partial x_{i_j}}(a) h_{i_1} \ldots h_{i_j}$,
$R_k = d^{(k)} f_c(h) = \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \frac{\partial^j f}{\partial x_{i_1} \ldots \partial x_{i_k}}(c) h_{i_1} \ldots h_{i_k}$ *and*
$h = (h_1, \ldots, h_n) = x - a$.

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Differential fo Vector-Valued Functions

Definition 29

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function of n variables. Let $x = (x_1, \ldots, x_n)$ denote a point of $\mathbb{R}^n$ and $f = (f_1, \ldots, f_m)$. We define the **matrix of partial derivatives of** f, denoted Df, to be the $m \times n$ matrix whose (i, j) entry is $\frac{\partial f_i}{\partial x_j}$. That is, $Df = \left(\frac{\partial f_i}{\partial x_j}\right)$. The matrix $Df(a) = \left(\frac{\partial f_i}{\partial x_j}(a)\right)$ is also called **Jacobian matrix of** f **at** a.

Differential fo Vector-Valued Functions

Theorem 30

Let $D \subset \mathbb{R}^n$ be open and $f : D \to \mathbb{R}^m$ be a vector-valued function of $x = (x_1, \ldots, x_n)$. If $f = (f_1, \ldots, f_m)$ is differentiable at a then the first-order partial derivatives of f exist at a and the differential of f at a is

$$df\,(dx) = Df(a)(dx) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix} \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}$$

For short,

$$df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix} \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}.$$

Differential fo Vector-Valued Functions

Theorem 31

If $f, g : D \to \mathbb{R}^m$ are differentiable at a then

$$d(f + \alpha g)(a) = df(a) + \alpha dg(a)$$

where α is a constant.

Theorem 32

Let $D \subset \mathbb{R}^n$ be open and $f : D \to \mathbb{R}^m$ be a function. If all first-order partial derivatives of f exist at a and are continuous at a, then the function f is differentiable at a.

Differential fo Vector-Valued Functions

Theorem 33 (Chain rule)

Let $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}^m$ be open. If $f : D_1 \to \mathbb{R}^m$ is differentiable at a and $g : D_2 \to \mathbb{R}^p$ is differentiable at $f(a) \in D_2$, then $k = g \circ f$ is differentiable at a and

$$d(g \circ f)(a) = dg\,(f(a))\,df(a).$$

If $M_{g \circ f}(a)$ is the Jacobian matrix of $g \circ f$ at a, $M_g\,(f(a))$ the Jacobian matrix of g at $f(a)$, and $M_f(a)$ is the Jacobian matrix of f at a, then

$$M_{g \circ f}(a) = M_g\,(f(a))\,M_f(a).$$

Gradient

Definition 23

For a real-valued function $f(x_1, x_2, \ldots, x_n)$, the **gradient of f at a point** a, denoted by $\nabla f(a)$, is the vector

$$\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \ldots, \frac{\partial f}{\partial x_n}(a) \right).$$

Theorem 24

Let $D \subset \mathbb{R}^n$ be open, and suppose $f : D \to \mathbb{R}$ is differentiable at $a \in D$. Then the directional derivative of f at a exists for all directions (unit vectors) $\mathbf{u}$ and, moreover, we have

$$D_{\mathbf{u}} f(a) = \nabla f(a) \cdot \mathbf{u}.$$

Gradient

Theorem 25

Let f be a continuously differentiable real-valued function, with $\nabla f \neq \mathbf{0}$. Then:

- *The value of $f(x)$ increases the fastest in the direction of ∇f. The maximum value of $D_{\mathbf{u}} f$ is $\|\nabla f\|$.*
- *The value of $f(x)$ decreases the fastest in the direction of $-\nabla f$. The minimum value of $D_{\mathbf{u}} f$ is $-\|\nabla f\|$.*

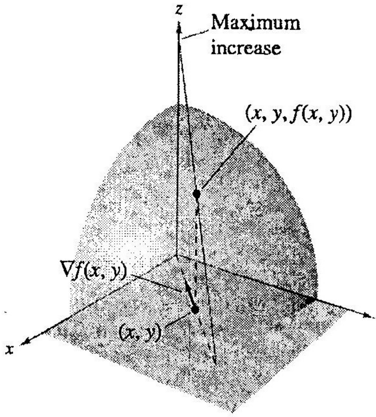

Figure: The maximum increase of f in the direction of $\nabla f(x, y)$ in the xy-plane

Gradient

Theorem 26

Let $D \subset \mathbb{R}^n$ be open, and $f : D \to \mathbb{R}$ be a function of class C^1. If a is a point on the level hypersurface $S = \{x \in D : f(x) = c\}$, then the vector $\nabla f(a)$ is perpendicular to S.

Differential fo Vector-Valued Functions

Definition 27

Let $D \subset \mathbb{R}^n$ be open and $f : D \to \mathbb{R}^m$ be a vector-valued function of n variables. Then f is said to be differentiable at a if there is a mapping $L_a : \mathbb{R}^n \to \mathbb{R}^m$ (called **the differential of f at** a denoted by $df_a = L_a$ or $df = L$ for short, that is $df(h) = L(h)$) such that

1. $L_a(\alpha x + y) = \alpha L_a(x) + L_a(y)$ for all $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.
2. $f(a + h) = f(a) + L_a(h) + o(\|h\|)$.

Theorem 28

If the function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a then there is only one mapping L_a and f is continuous at a.

Chain rules for functions of several variables

Theorem 19 (Chain rules for function of several variables)

Let $y = f(x_1, x_2, \ldots, x_n)$, where f is differentiable function of $x_i, i = 1, 2, \ldots, n$. If each $x_i, i = 1, 2, \ldots, n$ is a differentiable function of m variables $t_1, t_2, \ldots, t_m$, then y is a differentiable function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial y}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial y}{\partial x_i} \frac{\partial x_i}{\partial t_j}.$$

for $j = 1, 2, \ldots, m$.
In particular, if $x_i, i = 1, 2, \ldots, n$ is a function of a single variable t, then we have

$$\frac{dy}{dt} = \sum_{i=1}^{n} \frac{\partial y}{\partial x_i} \frac{dx_i}{dt}.$$

Chain rules for functions of several variables

Theorem 20 (Chain rule: Implicit Differentiation)

If the equation $F(x_1, x_2, \ldots, x_n, y) = 0$ defines y implicitly as a differentiable function of $x_i, i = 1, 2, \ldots, n$, then

$$\frac{\partial y}{\partial x_i} = -\frac{F_{x_i}(x_1, x_2, \ldots, x_n, y)}{F_y(x_1, x_2, \ldots, x_n, y)}, \quad F_y(x_1, x_2, \ldots, x_n, y) \neq 0$$

for $i = 1, 2, \ldots, n$.

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Directional Derivatives

Definition 21

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be a real-valued function and a be a point in $\mathbb{R}^n$. If $\mathbf{u} \in \mathbb{R}^n$ is any unit vector. Then, **the directional derivative of f at a in the direction of $\mathbf{u}$** is defined by

$$D_{\mathbf{u}} f = \lim_{t \to 0} \frac{f(a + t\mathbf{u}) - f(a)}{t}, \quad t \in \mathbb{R},$$

provided that the limit exists.

Theorem 22

Let $f : D \to \mathbb{R}$ and $a = (a_1, a_2, \ldots, a_n) \in D$. Suppose that the first partial derivatives of f exist and continue at a. Then the **directional derivative of** *f in the direction of a* **unit vector** *$\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is given by*

$$D_{\mathbf{u}} f(a) = \sum_{i=1}^{n} u_i f_{x_i}(a).$$

Differentials

Definition 14

Let $D \subset \mathbb{R}^n$ be open and $f : D \to \mathbb{R}$. The function f is said to be **differentiable at** a if there is a mapping $L_a : \mathbb{R}^n \to \mathbb{R}$ (called **differential of** f **at** a denoted by $df_a = L_a$ or $df = L$ for short, that is $df(h) = L(h)$), such that

1. $L_a(\alpha x + y) = \alpha L_a(x) + L_a(y)$ for all $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.
2. $f(a+h) = f(a) + L_a(h) + o(\|h\|)$.

Theorem 15

Let $D \subset \mathbb{R}^n$ be open and $f : D \to \mathbb{R}$ be a function. If the function f is differentiable at $a \in D$ then there is only one mapping L_a and f is continuous at a.

Theorem 16

Let $D \subset \mathbb{R}^n$ be open and $f : D \to \mathbb{R}$. If the function f is differentiable at a then all partial derivatives of function f exist at a and the differential of f at a is

$$df = L(h) = \frac{\partial f}{\partial x_1}(a)h_1 + \frac{\partial f}{\partial x_2}(a)h_2 + \cdots + \frac{\partial f}{\partial x_n}(a)h_n$$

where $h = (h_1, h_2, \ldots, h_n)$.

Note that if we denote the increments

$$h_i = \Delta x_i = dx_i, \quad i = 1, 2, \ldots, n$$

(called **the differential of** the independent variable $x_i, i = 1, 2, \ldots, n$, respectively), then

$$df = \frac{\partial f}{\partial x_1}(a)dx_1 + \frac{\partial f}{\partial x_2}(a)dx_2 + \cdots + \frac{\partial f}{\partial x_n}(a)dx_n$$

Differentials

and the increment $\Delta f = f(a + \Delta x) - f(a)$ of dependent variable f is

$$\Delta f = \frac{\partial f}{\partial x_1}(a)\Delta x_1 + \cdots + \frac{\partial f}{\partial x_n}(a)\Delta x_n + o(\|\Delta x\|).$$

If $\Delta x_i = dx_i, i = 1, 2, \ldots, n$ are small enough tending to zero, then Δf can be approximated by df.

Theorem 17

Let $D \subset \mathbb{R}^n$ be open and $f : D \to \mathbb{R}$. If all first-order partial derivatives of function f exist and are continuous at a, then the function f is differentiable at a.

Theorem 18

If a function is differentiable at a, then it is continuous at a.

Contents

Partial Derivatives

Definition 12

- A function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is said to be of **class** C^k if all its partial derivatives of order $\leq k$ are continuous.
- A function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is said to be of **class** C^∞ if f has continuous partial derivatives of all orders.
- A function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be of class C^k if each of component functions is of class C^k.
- A function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be of class C^∞ if each of component functions is of class C^∞.

Partial Derivatives

- Note that if the function if $\dfrac{\partial f}{\partial x_i}$ has a partial derivative with respect to x_j, we denote the partial derivative by

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j \partial x_i} = f_{x_i x_j}.$$

- The function obtained by differentiating f successively with respect to $x_{i_1}, x_{i_2}, \dots x_{i_r}$ at x is denoted by

$$\frac{\partial^k f}{\partial x_{i_r} \partial x_{i_{r-1}} \dots \partial x_{i_1}} = f_{x_{i_1} x_{i_2} \dots x_{i_r}} \quad \text{where} \quad i_1 + \cdots + i_r = k.$$

It is called a k**th-order partial derivative of** f.

Partial Derivatives

Theorem 13

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be a C^k function. Then

$$f_{x_{i_1} x_{i_2} \dots x_{i_r}}(x) = f_{x_{j_1} x_{j_2} \dots x_{j_r}}(x)$$

where $i_1 + i_2 + \cdots + i_r = j_1 + j_2 + \cdots + j_r = k$.

Contents

Continuity

Theorem 9

Let $f : D \subset \mathbb{R}^n \longmapsto \mathbb{R}^m$, where $f = (f_1, \ldots, f_m)$. Then f is continuous at $a \in D$ (respectively f is continous on D) if and only if its component functions $f_i : D \to \mathbb{R}$ are all continuous at a.

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Partial derivatives of a function of several variables

Definition 10

Let $f : D \subset \mathbb{R}^n \longmapsto \mathbb{R}$. If $y = f(x) = f(x_1, x_2, \ldots, x_n)$, then the **first partial derivative** of f with respect to $x_i, i \in \{1, 2, \ldots, n\}$, is defined by $f_{x_i}(x)$ or $\frac{\partial f}{\partial x_i}(x)$ and

$$f_{x_i}(x) = \frac{\partial}{\partial x_i} f(x) = \lim_{\Delta x_i \to 0} \frac{f(x_1, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, \ldots, x_n)}{\Delta x_i}$$

provided the limit exists.

Partial Derivatives

Theorem 11

Let f, g be two scalar-valued functions of n variables and let $x = (x_1, \ldots, x_n)$. If $f_{x_i}(x)$ and $g_{x_i}(x)$ exist, then

- $\dfrac{\partial(f + \lambda g)}{\partial x_i} x) = f_{x_i}(x) + \lambda g_{x_i}(x)$ *where λ is some constant*
- $\dfrac{\partial(fg)}{\partial x_i}(x) = f_{x_i}(x) g(x) + f(x) g_{x_i}(x)$
- $\dfrac{\partial(f/g)}{\partial x_i}(x) = \dfrac{f_{x_i}(x) g(x) - f(x) g_{x_i}(x)}{g^2(x)}$ *if $g(x) \neq 0$.*

Limit of a Function of Several Variables

Definition 3

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ and $a \in \bar{D}$. Then we say that the limit of $f(x)$ equals L as x approaches a, written as

$$\lim_{x \to a} f(x) = L,$$

if given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|f(x) - L\| < \varepsilon \quad \text{whenever} \quad \|x - a\| < \delta.$$

Theorem 4

If a limit exists, it is unique.

Limit of a Function of Several Variables

Theorem 5

Suppose that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist and that k is a scalar. Then

- $\lim_{x \to a}[f(x) \pm g(x)] = [\lim_{x \to a} f(x)] \pm [\lim_{x \to a} g(x)]$
- $\lim_{x \to a}[kf(x)] = k[\lim_{x \to a} f(x)]$
- $\lim_{x \to a}[f(x)g(x)] = [\lim_{x \to a} f(x)][\lim_{x \to a} g(x)]$
- $\lim_{x \to a}[f(x)/g(x)] = [\lim_{x \to a} f(x)]/[\lim_{x \to a} g(x)]$ *provided* $\lim_{x \to a} g(x) \neq 0$ *and both f and g are real-valued functions.*
- *If $f(x) \leq g(x)$ for all x, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$, where f and g are real-valued functions.*
- *If $\| \lim_{x \to a} f(x) - L\| \leq g(x)$ for all x and if $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} f(x) = L$.*

Limit of a Function of Several Variables

Note

To show that the limit does not exist, we need to show that the function approaches different values as x approaches a along different paths in $\mathbb{R}^n$.

Theorem 6

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ be a vector-value functions, $f = (f_1, f_2, \ldots, f_m)$ and $L = (L_1, L_2, \ldots, L_m)$. Then $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a} f_i(x) = L_i$ for $i = 1, 2, \ldots, m$.

Continuity

Definition 7

Let $f : D \subset \mathbb{R}^n \longmapsto \mathbb{R}^m$. We say that the function f is continuous at a point a in D if

$$\lim_{x \to a} f(x) = f(a).$$

We say that f is a continuous function on D if it is continuous at every point in its domain D.

Theorem 8 (Algebraic properties)

Let $f, g : D \subset \mathbb{R}^n \to \mathbb{R}^m$ be continuous vector-value function and let $\alpha \in \mathbb{R}$ be a scalar. Then

- *$f + \alpha g$ and fg are continuous.*
- *If both f and g are real-valued functions and if $g(x) \neq 0$, then f/g is continuous.*

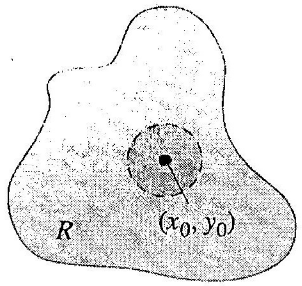

Figure: Interior point

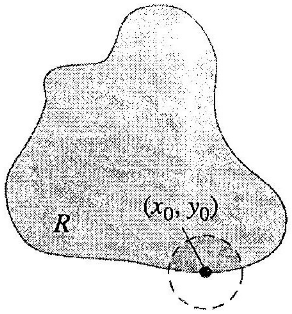

Figure: Boundary point

- A region R is **open** if it is a subset of its interior. That is, $R \subset \overset{\circ}{R}$
- A region R is **closed** if it contains its entire boundary. That is, $\partial R \subset R$.
- The closure of R is denoted by $\overline{R}$ and defined by

$$\overline{R} = \overset{\circ}{R} \cup \partial R$$

Level curve, level surface and level hypersurfce

Definition 2 (Level curve, level surface and level hypersurfce)

The set of points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ where a function of n independent variables has a constant value $f(x_1, x_2, \ldots, x_n) = c$ is called a **level hypersurface** of f.
In particular,

- the set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f.
- the set of points in the space where a function $f(x, y, z)$ has a constant value $f(x, y, z) = c$ is called a **level surface** of f.
- Note that if $n \geq 4$, the set of points satisfying the equation $f(x_1, x_2, \ldots, x_n) = c$ is called **level hypersurface**.

Example of level curves

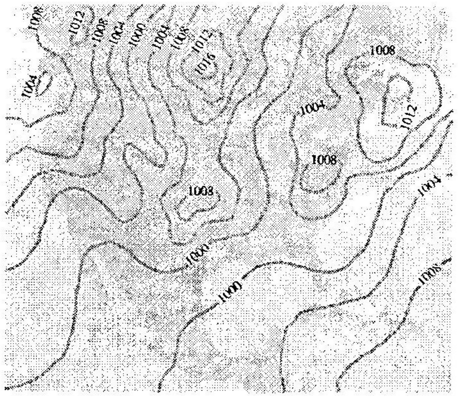

Figure: Level curves show the lines of equal pressure (isobars) measured in millibars

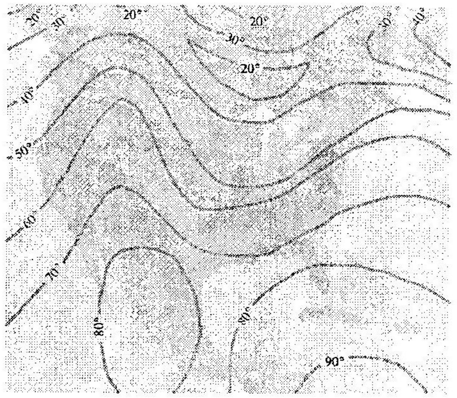

Figure: Level curves show the lines of equal temperature (isotherms) measured in degree Fahrenheit.

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Note

A **real-valued function** on subset D of $\mathbb{R}^n$ is a function whose range is $\mathbb{R}$. That is,

$$f : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}; (x_1, x_2, \dots, x_n) \longmapsto y = f(x_1, x_2, \dots, x_n)$$

Special cases for $n = 2$ and $n = 3$ will be mainly concerned since they help to visualise their geometrical meaning.

Some Operations on

Let $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. We define

- Addition:
$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$
- Scalar multiplication:
$$\alpha x = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$$
- Inner product:
$$x \cdot y = \langle x, y \rangle = xy = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

- In this study we use only **Euclidean Norm**, that is if $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, then norm of x is defined by
$$\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$
- If $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, then **norm of the difference** x **and** y (or **Euclidean distance between** x **and** y) is defined by
$$\|x - y\| = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}.$$

- Let $x_0 \in \mathbb{R}^n$ and let $\epsilon > 0$. A **neighbourhood** or **ϵ-neighbourhood** about x_0 is denoted and defined by
$$N_\epsilon(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < \epsilon\}.$$
- A point x_0 in R is called an **interior point** of R if there exists an ϵ-neighbourhood about x_0 that lies entirely in R. That is,
$$x_0 \in N_\epsilon(x_0) \subset R.$$
The **interior** of R, denoted by $\overset{\circ}{R}$ or $\text{int}(R)$, is the set of all interior points of R.
- A point x_0 is a **boundary point** of R if every neighbourhood about x_0 contains points inside R and points outside R.
$$\forall \epsilon > 0 : N_\epsilon(x_0) \cap R \neq \emptyset \quad \text{and} \quad N_\epsilon(x_0) \cap R^c \neq \emptyset$$
The **boundary** of R, denoted by ∂R or $b(R)$, is the set of all boundary points of R.

Lagrange Multipliers

Theorem 56 (Second derivative test for constrained local extremum)

$f, g_1, \ldots, g_k : D \to \mathbb{R}$ be C^2 functions where $D \subset \mathbb{R}^n$ is open and $k < n$. Denote $(\lambda; x)$ and so-called ***Lagrange function*** *$L(\lambda; x) = f(x) - \sum_i^k \lambda_i (g_i(x) - c_i)$. Suppose there is an $x \in D$ such that*

$$\det \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(a) & \cdots & \frac{\partial g_1}{\partial x_k}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial x_1}(a) & \cdots & \frac{\partial g_k}{\partial x_k}(a) \end{pmatrix} \neq 0$$

where $h_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_i}(a) - \sum_{b=1}^k \lambda_b \frac{\partial^2 g_b}{\partial x_j \partial x_i}(a)$. Let d_j be jth principal minor of $HL(\lambda; a)$.

Lagrange Multipliers

We calculate the sequence of $n - k$ numbers

$$(s) : (-1)^k d_{2k+1}, (-1)^k d_{2k+2}, \ldots, (-1)^k d_{k+n}$$

1. If the sequence in (s) consists entirely of positive numbers, then $f(a)$ is a local minimum subject to the constraints $g_i x) = c_i$ for all $i = 1, 2, \ldots, k$.
2. If the sequence in (s) begin with a negative number and thereafter alternates in sign, then $f(a)$ is a local maximum subject to the constraints $g_i(x) = c_i$ for all $i = 1, 2, \ldots, k$.
3. If neither case 1 nor case 2 holds, then f has a constrained saddle point at a.

References

1. R. Larson and B. Edwards, *Multivariable Calculus*, Ninth Edition, Brooks/Cole, Cengage Learning, 2010.
2. S. T. TAN, *Multivariable Calculus*, Brooks/Cole, Cengage Learning, 2010.